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# The eikonal phase of supersymmetric Coulomb partners

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**Abstract.** We investigate the eikonal phase and its systematic corrections for the two supersymmetric Coulomb partners $V_1$ and $V_2$ derived by Amado. Apart from a constant shift of $-\pi$ for $V_1$ and $-2\pi$ for $V_2$, the eikonal phase decay to the eikonal phase of the Coulomb potential as $1/kb$. For the potential $V_2$, which is phase equivalent to the Coulomb potential, this result is only valid at $b \simeq 0$ and asymptotically; in the intermediate range, it constitutes a lower limit.

## 1. Introduction

There is no general argument allowing us to conjecture that two phase equivalent potentials should have the same eikonal phase, except for an asymptotically large incident energy. On the other hand, for two supersymmetric (SS) partners differing by a single bound state, we may expect a similar structure of the eikonal phase at least at large impact parameter.

In this respect, the Coulomb potential is of particular interest. It is well known in this case that the eikonal approximation yields the exact result. The systematic corrections introduced by Wallace [1] vanish at all orders for $1/r$ potentials. The expansion obtained by Waxman *et al* [2] has a similar property: beyond the eikonal phase (zero-order), odd higher-order terms vanish, whereas even-order terms diverge at zero impact parameter but decrease very rapidly with increasing incident energies and impact parameters. Consequently it is interesting to study to what extent such properties are preserved in SS partners of the Coulomb potential.

Note that both expansions use the WKB approximation as a dynamical model. The Wallace expansion, on the other hand, includes also higher-order WKB terms. Thus this last converges more rapidly towards the exact result. However, owing to its simplicity and to the possibility of summing the series by estimating the WKB phase, we shall rely on the expansion of Waxmann *et al* [2].

The purpose of the present work is to study the eikonal phase of the two SS potentials $V_1$ and $V_2$ derived by Amado [3] for the Coulomb case following the method of Baye [4]. We shall show that much of the original eikonal phase expansion is preserved in the SS Coulomb partners. Apart from a constant shift of $-\pi$ and $-2\pi$ for $V_1$ and $V_2$, respectively, the origin of which has to be understood from the generalized Levinson theorem [5, 6], the difference with respect to the Coulomb potential eikonal phase is proportional to $1/kb$.

The results we obtain for the Coulomb potential are indeed valid for a large class of potentials. This follows from quite general arguments developed by Khare and Sukhatme [7] and Amado *et al* [8]. These arguments will be briefly recalled in the conclusions.

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The paper is organized as follows. In section 2 we recall the expressions for $V_1$ and $V_2$ together with some of their properties. Their eikonal phase and the systematic expansion of the correction terms are calculated in section 3. Conclusions are drawn in section 4.

## 2. Supersymmetric Coulomb partners

We consider a particle of mass $m$ moving in a Coulomb potential $V(r) = -e^2/r$. After partial wave expansion, we recall that the radial Schrödinger equation reads

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{e^2}{r} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]\psi(r) = E\psi(r). \tag{1}$$

To some extent, it is useful to introduce the dimensionless variable

$$x = \frac{me^2}{\hbar^2(\ell+1)}r$$

in which case (1) becomes

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2\frac{(\ell+1)}{x} + \frac{\ell(\ell+1)}{x^2}\right]\psi(x) = \varepsilon\psi(x) \tag{2}$$

where

$$\varepsilon = \frac{2}{m}\left[\frac{\hbar(\ell+1)}{e^2}\right]^2 E.$$

The two SS potentials derived by Amado [3] are written ($i = 1, 2$)

$$V_i(r) = -\frac{e^2}{r} + \Delta V_i(r) \qquad V_i(x) = -2\frac{(\ell+1)}{x} + \Delta V_i(x). \tag{3}$$

For the first case, $i = 1$, we have

$$\Delta V_1(r) = \frac{\hbar^2}{m}\frac{(\ell+1)}{r^2} \qquad \Delta V_1(x) = 2\frac{(\ell+1)}{x^2}. \tag{4}$$

The second case $\Delta V_2$ has a more complicated $\ell$-dependence. We quote only its $x$-dependence:

$$\Delta V_2(x) = 2\alpha(x)\left[-\frac{2(\ell+1)}{x} + \alpha(x) + 2\right] \tag{5}$$

where

$$\alpha(x) = \frac{2\ell+3}{x}[{}_1F_1(1, 2\ell+4; 2x)]^{-1} \tag{6}$$

so that

$$\Delta V_2 = -2(2\ell+3)\frac{\mathrm{d}}{\mathrm{d}x}[x\,{}_1F_1(1, 2\ell+4; 2x)]^{-1}. \tag{7}$$

Note that $V_2$ is phase equivalent to the Coulomb potential, whereas $V_1$ is not. Since the $\ell$-dependence of $V_2$ is far from being simple, we shall first discuss a few of its properties.

It is easy to show that $\Delta V_2(x) > 0$, and that furthermore

$$\Delta V_2(x) \leqslant \frac{2(2\ell+3)}{x^2}. \tag{8}$$

By using (5) and (6), this inequality can be written

$$2\ell + 3 + 2x\,{}_1F_1 \leqslant {}_1F_1^2 + 2(\ell+1)\,{}_1F_1 \tag{9}$$

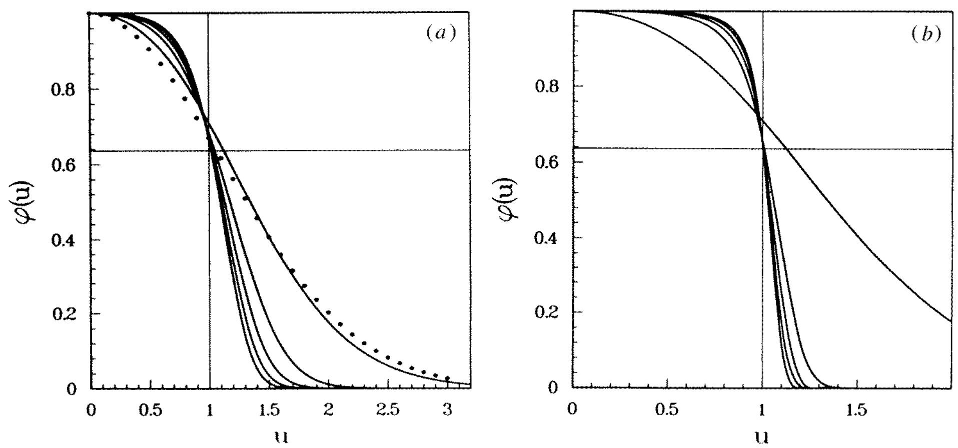


**Figure 1.** Function $\varphi(\ell, u)$ versus $u$ for $(a)$ $\ell = 0, 5, 10, 15$ and 20 and $(b)$ $\ell = 0, 50, 100, 150$ and 200. The slope of $\varphi(\ell, u)$ gets steeper as $\ell$ increases. The asymptotic value of $\varphi(\infty, 1)$ is also indicated. In $(a)$ the full circles represent the fit to $\varphi(0, u)$ by the $e^{-0.4u^2}$.

and it is proved order by order in the powers of $x$. For $\ell = 0$, the explicit expression for $\Delta V_2$ , is given from (5) and (6) by

$$\Delta V_2(x, \ell = 0) = \frac{16x[e^{2x}(x-1) + x + 1]}{[e^{2x} - 1 - 2x - 2x^2]^2}. \tag{10}$$

The higher angular momentum values lead to much more complicated expressions. In this respect, a graphic representation of $\Delta V_2$ will be useful. We start from (7) and write

$$\frac{\Delta V_2 x^2}{2(2\ell+3)} = \frac{{}_1F_1 + x\,{}_1F_1'}{({}_1F_1)^2} = \frac{N}{D^2} = \varphi(\ell, x) \tag{11}$$

where ${}_1F_1'$ denotes the derivative with respect to $x$. By substituting $x = (\ell+2)u$, we have

$$N = 1 + 2u + 3u^2\frac{2\ell+4}{2\ell+5} + 4u^3\frac{(2\ell+4)^2}{(2\ell+5)(2\ell+6)} + \cdots$$
$$D = 1 + u + u^2\frac{2\ell+4}{2\ell+5} + u^3\frac{(2\ell+4)^2}{(2\ell+5)(2\ell+6)} + \cdots.$$

As shown in figures 1$(a)$ and $(b)$, as $\ell$ increases $\varphi(\ell, u)$ approaches the step function $\Theta(1-u)$. This can be inferred from the following arguments. In the expansion for $N$, for $0 \leqslant u < 1$, the generic term

$$\frac{nu^{n-1}(2\ell+4)^{n-2}}{(2\ell+5)\ldots(2\ell+3+n)} \tag{12}$$

is dominated by $nu^{n-1}$, which is also its limit for $\ell \to \infty$. Thus, $N \to \frac{1}{(1-u)^2}$. Similarly we have $D \to \frac{1}{1-u}$.

Consequently, for $u < 1$ as $\ell \to \infty$

$$\varphi(\ell, u) = 1 \qquad \varphi(\ell, u)' = 0. \tag{13}$$

For $u > 1$, we first make use of the Kummer transform

$${}_1F_1(1, \nu, \nu u) = e^{\nu u}\,{}_1F_1(\nu - 1, \nu, -\nu u) = e^{\nu u}\frac{\nu - 1}{(\nu u)^{(\nu-1)}}\gamma(\nu - 1, \nu u) \tag{14}$$

where $\nu = 2\ell + 4$ and $\gamma$ in the incomplete $\gamma$ function [9] with the following limits

$$\begin{aligned} \lim \nu \to \infty \qquad & \gamma(\nu - 1, \nu u)/\Gamma(\nu - 1) = 1 \qquad u > 1 \\ & = \tfrac{1}{2} \qquad u = 1. \end{aligned} \tag{15}$$

Then it is easy to show that

$$\varphi(\ell, u) = \frac{(\nu u)^{(\nu-1)}[(\nu u)^{(\nu-1)} + \mathrm{e}^{\nu u}\gamma(\nu - 1, \nu u)(2 + \nu(u - 1))]}{\mathrm{e}^{2\nu u}(\nu - 1)\gamma^2(\nu - 1, \nu u)}. \tag{16}$$

For $u > 1$, we obtain

$$\varphi(\ell, u) \approx \frac{(u - 1)}{u}\frac{(\nu u \mathrm{e}^{-u})^{\nu}}{\Gamma(\nu)}. \tag{17}$$

It shows that $\varphi(\ell, u)$ tends only asymptotically to a step function. For finite $\ell$ its exponential behaviour around $u = 1$ is well illustrated by the curves of figures 1($a$) and ($b$). Finally, by setting $u = 1$ in (16), and letting $\ell \to \infty$, we get $\varphi(\infty, 1) = 2/\pi$. Again this value is reached asymptotically (see figures 1($a$) and ($b$)).

## 3. Eikonal expansion for $V_1$ and $V_2$

To study the eikonal phase and its corrections for $V_1$ and $V_2$, we use the systematic series expansion obtained by Waxmann *et al* [2] from the WKB approximation taken as a dynamical model, as stated in the introduction. We recall that the eikonal phase is expressed as a function of the impact parameter $b$ and its conjugate variable $k$, the incident momentum. It yields

$$\chi(b) = -\sum_{n=0}^{\infty} \frac{m^{n+1}}{\hbar^{2n+2}k(n+1)!}\left\{\left(\frac{b}{k}\frac{\mathrm{d}}{\mathrm{d}b} - \frac{\mathrm{d}}{\mathrm{d}k}\right)\frac{1}{k}\right\}^n \int_{-\infty}^{\infty} V^{n+1}(r)\,\mathrm{d}z. \tag{18}$$

Here the differentiations with respect to $b$ and $k$ are carried out at fixed $k$ and $b$, respectively. Note also that for $\ell$-dependent potentials, the semiclassical substitution $\ell = kb - \frac{1}{2}$ brings no contribution from the product $kb$. Consequently the $\ell$ dependence can be taken out of the operator.

The $n$th-order term can be written as the Abel transform of the $(n+1)$th power of the potential:

$$\chi_n = -2k\left(\frac{m}{\hbar^2 k^2}\right)^{n+1}\frac{1}{b^{2n}(n+1)!}(b^2(1 + b\partial b))^n \int_b^{+\infty} \frac{V(r)^{n+1} r\,\mathrm{d}r}{\sqrt{r^2 - b^2}}. \tag{19}$$

For the bare Coulomb potential, the odd $n$ terms vanish to all orders; the even $n$ contributions up to the fourth order are given by $(kb = q)$

$$\begin{aligned} \chi_0 &= 2y\ln(q) + \text{constant} \\ \chi_2 &= -\frac{1}{3}y^3\frac{1}{q^2} \\ \chi_4 &= \frac{1}{10}y^5\frac{1}{q^4} \end{aligned} \tag{20}$$

where use is made of $y = \frac{m}{\hbar^2}\frac{e^2}{k}$. The very simple structure of $V_1$ leads to a compact expression:

$$\chi_n^{(1)} = -\frac{(-2)^n\sqrt{\pi}}{(n+1)!}\sum_{p=0}^{n+1} C_{n+1}^p \frac{\Gamma(n-p/2+\frac{1}{2})}{\Gamma(n+1-p/2)} \times\left[\prod_{j=1}^{n}\left(j-\frac{p}{2}\right)\right](-y)^p(\ell+1)^{n+1-p}q^{-2n-1+p} \tag{21}$$

which is valid for $n \geqslant 1$.

Here the coefficients $C_{n+1}^p$ are the binomial coefficients. Note that for powers $n > 1$, $\chi_n^{(1)}$ involves powers of $\Delta V_1$ and mixed contributions including the Coulomb potential.

The corrections to $\chi_n$ due to $\Delta V_1$ are listed below up to $n = 2$. We get successively

$$\begin{aligned}
\Delta(\chi_0^{(1)}) &= -\pi\left(1+\frac{1}{2q}\right)\\
\Delta(\chi_0^{(1)}) + \Delta(\chi_1^{(1)}) &= -\pi\left(1-\frac{1}{2q^2}-\frac{1}{8q^3}\right) - y\left(\frac{2}{q}+\frac{1}{q^2}\right)\\
\Delta(\chi_0^{(1)}) + \Delta(\chi_1^{(1)}) + \Delta(\chi_2^{(1)}) &= -\pi\left(1+\frac{5}{8q^3}+\frac{3}{8q^4}+\frac{1}{16q^5}\right)\\
&\quad -y\left(\frac{2}{q}-\frac{1}{q^2}-\frac{2}{q^3}-\frac{1}{2q^4}\right).
\end{aligned} \tag{22}$$

This result establishes clearly that besides a shift of $-\pi$, the eikonal phase of $V_1$ differs from the original Coulomb phase by a contribution which vanishes only asymptotically. This last diverges at $b = 0$. Among the higher-order terms, we distinguish two types of contributions. Those arising from $\Delta V_1$ only show the interesting property of a recurrent cancellation. To be explicit, the $-\pi/q$ of $\Delta\chi_0^{(1)}$ is cancelled by the lowest contribution to $\Delta\chi_1^{(1)}$, and such a cancellation propagates from one order to the next, so that the partial summation to the $n$th order diverges at least like $1/q^{n+1}$.

The terms proportional to powers of $y$ involve the Coulomb potential. They decay faster as the energy is increasing due to the extra $1/k$ factor in $y$. Even powers of $y$ do not appear for the very same reason as for the Coulomb potential. The recurrent cancellation does not occur for these terms.

For the sake of comparison, it is useful to calculate the WKB phase shift for $V_1$. By using standard techniques we end up with

$$\delta_{\mathrm{WKB}} = -\frac{\pi}{2} + q\arctan\frac{y}{q} - (q+1)\arctan\frac{y}{q+1} + \frac{y}{2}\log\frac{q^2+y^2}{(1+q)^2+y^2} + \delta_{\mathrm{coul}} \tag{23}$$

where

$$\delta_{\mathrm{coul}} = y\ln(q) + \sum_{n\ \mathrm{even},\geqslant 2}(-)^{n/2}\frac{1}{n(n+1)}\frac{y^{n+1}}{q^n} + \text{constant}. \tag{24}$$

Remembering that

$$\chi(b) = 2\delta(\ell)$$

we can check this result against the first few terms of (22). If we only keep the contribution from $\Delta V_1$, setting $y = 0$, we obtain a shift of $-\pi/2$. No trace of the $1/q$ divergences at $b = 0$ remains, which indicates that the convergence domain of the series expansion (18) is given by $b > k^{-1}$. The expressions (22) and (23) show also that as $q \to \infty$, $\delta_{\mathrm{WKB}} \to -\pi/2$.

The complicated $\ell$- and $r$-dependences of the potential $V_2$ prevent us from giving a compact expression for the eikonal phase. Consequently we shall present and discuss a couple of limits.

From the inequality (8), it is straightforward to show that

$$\Delta\chi_0^{(2)} \geqslant -2\pi - \frac{2\pi}{kb}. \tag{25}$$

This is not enough, however, to ensure that asymptotically $\chi_0^{(2)}$ reduces to a $2\pi$ shift proving the phase equivalence of $V_2$. Numerical estimates of $V_2$ indicate that as $b \simeq 0$, we have

$$\Delta V_2(r) \approx \frac{(2\ell+3)\hbar^2}{m}\frac{1}{r^2}\mathrm{e}^{-\omega^2 r^2} \tag{26}$$

with $\omega \simeq 0.1\frac{me^2}{\hbar^2}$. This leads to

$$\Delta\chi_0^{(2)} \approx -2\pi\left[1+\frac{1}{kb}\right](1-\phi(\omega b)). \tag{27}$$

The estimate of $\Delta\chi_1^{(2)}$ shows that the recurrent cancellation observed for $\Delta V_1$ does not occur for $\Delta V_2$. As $b \to \infty$, the discussion of the preceeding section together with the shape of $\varphi(\ell, u)$ displayed in figures 1($a$) and ($b$) suggest that

$$\Delta V_2(r) \approx \frac{(2\ell+3)\hbar^2}{m}\frac{1}{r^2}\Theta(r_0 - r) \tag{28}$$

where $r_0$ actually depends on $\ell$: $r_0 = (\ell+1)(\ell+2)\frac{\hbar^2}{me^2}$. A straightforward calculation leads to

$$\Delta\chi_0^{(2)} \approx -4\frac{(kb+1)}{kb}\arccos\frac{b}{r_0}. \tag{29}$$

As $\frac{b}{r_0} \to 0$ we again obtain the same estimate as above.

In order to put our estimate of $\Delta(\chi_0^{(2)})$ on firmer ground, let us start again from the general expression

$$\Delta(\chi_0^{(2)}) = -2\frac{m}{\hbar^2 k}\int_b^{+\infty}\frac{\Delta V_2(r)r}{\sqrt{r^2-b^2}}\,\mathrm{d}r \tag{30}$$

or equivalently

$$\Delta(\chi_0^{(2)}) = -\frac{4\ell+6}{kb}\int_1^{+\infty} G(r)\frac{1}{r\sqrt{r^2-1}}\,\mathrm{d}r \tag{31}$$

where

$$\begin{aligned} G(r) &= \left.\frac{[x\,{}_1F_1(1, 2\ell+4, 2x)]'}{{}_1F_1(1, 2\ell+4, 2x)^2}\right|_{x=Zr} \\ Z &= \frac{m}{\hbar^2}\frac{e^2}{k}\frac{kb}{kb+\frac{1}{2}}. \end{aligned} \tag{32}$$

When the variable $Z$ tends to zero, which happens for infinite values of $k$ or for vanishing values of $kb$, the function $G(r)$ tends to $G(0) = 1$. The dominated convergence theorem states that

$$\Delta(\chi_0^{(2)}) \mapsto -2\pi\frac{kb+1}{kb} \simeq -2\pi\left(1+\frac{1}{kb}\right). \tag{33}$$

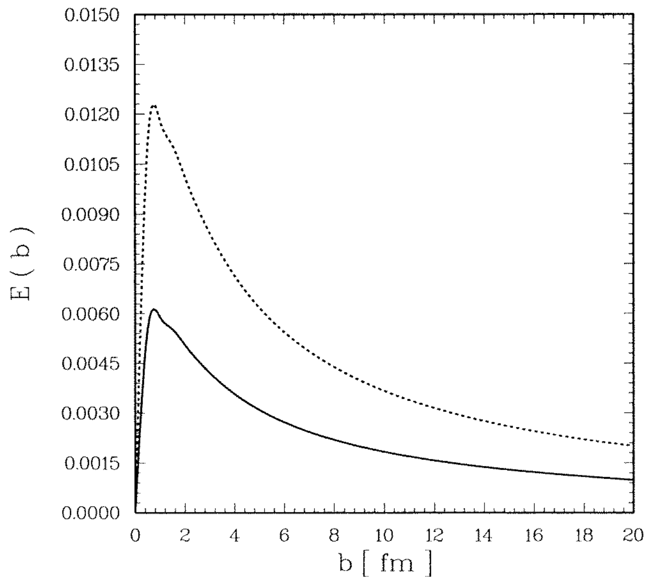


**Figure 2.** Relative difference $E(b)$ (equation (34)) between the asymptotic and exact value of $\chi_0^{(2)}(b)$ (see text). The full and broken curves correspond to the 'proton' and the '$^4$He' cases, respectively. The $^4$He curve has been divided by 4.

As for the case of $V_1$, it would be very desirable to give the WKB value of the phase shift. This can be achieved analytically only in the asymptotic domain as $b \to \infty$. This is, however, sufficient for the present discussion. Following standard techniques as previously we find $\Delta(\delta_\ell) = -\pi$. Again this proves, at least asymptotically, the phase equivalence of $V_2$.

Finally, in order to check the behaviour of $\chi_0^{(2)}(b)$ over intermediate values of $b$, we display in figure 2 the relative difference between the exact value $\chi_0^{(2)}(b)$, obtained numerically, and the approximate (asymptotic) value $\chi_a^{(2)}(b) = -2\pi(\frac{kb+1}{kb})$, namely

$$E(b) = \frac{-2\pi(kb+1) - kb\chi_0^{(2)}(b)}{-2\pi(kb+1)}. \tag{34}$$

The result depends on the scale parameter $\frac{me^2}{\hbar^2}$. We shall consider two examples taken from nuclear physics. The length unit is the fm, and we fixed $k = 1\ \mathrm{fm}^{-1}$. The calculations have been performed for $\frac{me^2}{\hbar^2} = 0.0347\ \mathrm{fm}^{-1}$ and $0.2776\ \mathrm{fm}^{-1}$; it corresponds to the scattering of a proton and a $^4$He nucleus by a Coulomb potential fixed in space (see equation (1)). The general behaviour of $E(b)$ is to exhibit a maximum at low values of the impact parameter. The decay towards the asymptotic value at large $b$ is quite slow.

We find $E(b) \geqslant 0$, which confirms that $\chi_a^{(2)}(b)$ is a lower limit, as expected from (8).

## 4. Conclusions

In this work we have addressed the question of the eikonal phase of SS partners of the Coulomb potential. We have investigated the two cases $V_1$ and $V_2$ derived by Amado [3], $V_2$ being phase equivalent to the Coulomb potential by construction. We find that apart from a constant shift of $-\pi$ for $V_1$ and $-2\pi$ for $V_2$, respectively, the eikonal phase reaches asymptotically the original Coulomb eikonal phase. The rate of convergence to the Coulomb result is proportional to $1/kb$. It is interesting to note that asymptotically the eikonal phase reaches the results predicted by the generalized Levinson theorem [5, 6].

In the case of $V_1$, we observe an interesting recurrent cancellation among the successive contributions so that summing the eikonal series expansion up to order $n$ for $\Delta V_1$ yields in the leading order a divergence proportional to $(1/kb)^n$. This result brings a limitation to the convergence domain of the series expansion (18).

Although derived for the specific case of the Coulomb potential, our findings are in fact valid for a very large class of potentials. It has been shown by Khare and Sukhatme [7] that a SS partner is generated by a one-parameter family of potentials all being phase equivalent. It is then obvious that our results extend to any potential belonging to the 'Coulomb' family.

Futhermore, according to the work of Baye [4], the difference between the original potential and its SS partners presents a general feature, namely a $1/r^2$ behaviour near the origin and a rapid decay as $r$ increases. The same argument has been given by Amado *et al* [8], who showed that the SS partners behave like $(\ell+1)(\ell+2)/r^2$ near the origin, irrespective of the shape of the original potential and its number of bound states. This ensures a common behaviour of the eikonal phase as $b \to 0$. At the other extreme, as $b \to \infty$, it is dominated by the centrifugal barrier for any finite-range potential.

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